Mathematics for engineers

A bridge course from school to university

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Contents

1 Mathematical notation

1.1 Sets

Sets are collections of elements described by some property P. The standard notation for sets is

$$A = \{x : x \, \text{has property} \, P\},$$

where one reads: A consists of all x such that x has property P. Some sets as the real numbers or the whole numbers have special symbols. See Table 1.

Symbol	Description
$\mathbb{R}$	real numbers
$\mathbb{Z}$	whole numbers
$\mathbb{N}$	natural numbers
$\mathbb{N}_0$	natural numbers containing 0
$\mathbb{Q}$	rational numbers
$\mathbb{C}$	complex numbers

Table 1: Notation of certain sets.

The most important sets of this lecture are sub-sets of the real numbers, called intervals. An interval is a set of numbers characterized by their left and right "boundary". For example

$$[a, b] = \{x \in \mathbb{R} : \quad a \le x \le b\}$$

which we read as the closed interval a, b. Closed means that it containes a and b. An open interval does not contain the boundary points, i.e.

$$(a, b) = \{x \in \mathbb{R} : \quad a < x < b\}.$$

One cal also consider the half-open cases

$$[a, b) = \{x \in \mathbb{R} : \quad a \le x < b\}$$

and

$$(a, b] = \{x \in \mathbb{R} : \quad a < x \le b\}.$$

We also denote $\mathbb{R} = (-\infty, +\infty)$. All numbers smaller than a would be denoted by $(-\infty, a)$, all numbers smaller or equal to a by $(-\infty, a]$. Similarly, one defines the sets of all numbers larger that or larger or equal to a given number. If we have the situation that we describe x as having either the property $x \ge a$ or $x \le -a$ for a given $a \ge 0$, then we can write

$$\{x \in \mathbb{R} : \quad x \ge a \quad \text{or} \quad x \le a\}$$

which is the same as

$$x \in (-\infty, -a] \cup [a, +\infty).$$

Exercise 1.1. *Write down examples. Do you understand the notation?*

1.1.1 Operations on sets

Definition 1.1 (Intersection/Union/Difference). *We denote by $A \cap B$ the intersection of A and B which means that $A \cap B$ contains elements that are in A as well as in B. By $A \cup B$, we denote the union of the two sets A and B which means that $A \cup B$ contains elements that are either in A or in B. With $A \setminus B$, we denote finally the difference of A and B that means that $A \setminus B$ contains all elements in A that are not in B.*

Remark 1.1. *Of course the intersection and union is not limited to a finite number. If one has a family of sets $\{A_i : i \in I\}$ indexed by a countable or uncountable set I one can consider the sets $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$. For Example:*

$$\mathbb{R} = \bigcup_{n \in \mathbb{N}} [-n, n], \qquad \{0\} = \bigcap_{n \in \mathbb{N}} \left[-\frac{1}{n}, \frac{1}{n}\right].$$

1.2 Sums and products

To shorten expressions as

$$1 + 2 + 3 + 4 + \cdots + n, \tag{1.1}$$

we introduce the $\sum$-notation. We rewrite (1.1) as

$$1 + 2 + \cdots + n = \sum_{k=1}^{n} k.$$

In general, we have expressions as

$$\sum_{k=k_0}^{n} a_k = a_{k_0} + a_{k_0+1} + \cdots + a_{k_0+n}.$$

We read the expression as: Sum over a_k (a sub k) from $k = k_0$ to n.
The notation allows us to express long formulas in a compact form. For instance,

$$(a+b)^n = \binom{0}{n} a^n + \binom{1}{n} a^{n-1} b + \cdots + \binom{n-(n-1)}{n} a b^{n-1} + \binom{n}{n} b^n$$

can be expressed as

$$(a+b)^n = \sum_{k=0}^{n} \binom{k}{n} a^{n-k} b^k.$$

The symbol $\binom{k}{n}$ (read n choose k) is defined by

$$\binom{k}{n} = \frac{n!}{k!(n-k)!}, \quad \text{where} \quad m! = m(m-1)(m-2)\ldots 2 \cdot 1.$$

The number $\binom{k}{n}$ is called a binomial coefficient.

Exercise 1.2. *Convince yourself, that the formula (shift of the summation index)*

$$\sum_{k=k_0}^{n} a_k = \sum_{k=k_0-l}^{n-l} a_{k+l}$$

is true. (consider examples)

To write products, we introduce the $\prod$-notation. Similar to the $\sum$-notation, we can rewrite

$$1 \cdot 2 \cdot \dots \cdot n$$

as

$$1 \cdot 2 \cdot \dots \cdot n = \prod_{k=1}^{n} k.$$

2 Some simple formulas

Consider an arithmetic progression $a_0 \in \mathbb{R}$, $a_{n+1} = a_n + d$ for $n \in \mathbb{N}_0$, where $d \in \mathbb{R}$. By definition, two successive a_k of the arithmetic progression have the constant difference d, i.e. $a_k - a_{k-1} = d$ for all $k \in \mathbb{N}$. This also gives the formula

$$a_n = a_0 + (n-1)d, \quad n \geq 1.$$

Examples for arithmetic progressions are

$$\begin{aligned} &1, 2, 3, 4, 5, \dots \quad && d = 1,\, a_0 = 1, \\ &1, 3, 5, 7, 9, \dots \quad && d = 2,\, a_0 = 1, \text{and} \\ &5, 10, 15, 20, 25, \dots \quad && d = 5,\, a_0 = 5. \end{aligned}$$

Let us figure out a sum formula for arithmetic progressions. To get an idea that might generalise to the general case, let us first consider the sum of the first n numbers. A trick, that is often attributed to young Gauß. We list the numbers once from 1 to n and once from n to 1:

$$\begin{matrix} 1 & 2 & 3 & 4 & \dots & n \\ n & n-1 & n-2 & n-3 & \dots & 1 \end{matrix}$$

Then one realises that the sum of the two numbers per column is constant $n+1$. Since we have n times $n+1$ which we have to divide by 2 then. Thus, it is clear that

$$1 + 2 + 3 + 4 + \dots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}. \tag{2.1}$$

Does this idea apply to other arithmetic progressions? Let us consider

$$\begin{matrix} 1 & 3 & 5 & 7 & \dots & 2n-1 \\ 2n-1 & 2n-3 & 2n-5 & 2n-7 & \dots & 1 \end{matrix}$$

and the columnwise sum is $2n$. Again, we overcount by a factor of 2, and thus get

$$1 + 3 + 5 + \dots + 2n - 1 = \sum_{k=1}^{n} (2k-1) = n^2.$$

With the same method, we can prove

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (a_0 + (k-1)d) = \frac{n(2a_0 + (n-1)d)}{2}. \tag{2.2}$$

Exercise 2.1. *Verify that* (2.2) *gives* (2.1) *for* $a_0 = 1$ *and* $d = 1$.

3 Arithmetic of algebraic expressions

In this section, we will learn how to deal with expressions like

$$\frac{ax^2+xy^2-5by}{x-8}, \quad \frac{ab^3+a}{x^2-y^2}, \quad \frac{a^3-b^3}{a-b}, \quad \frac{abc+(abc)^2-17}{15x+8x^2+6}.$$

We will investigate how to simplify them, add and subtract, and multiply and divide them. We will assume that the variables assume real numbers. Most of the considerations carry over to complex numbers.

Addition of simple algebraic expressions. Consider the two simple algebraic expressions ax^2+5x and $2x^3+5x^2-12$. Considering

$$(ax^2+5x)+(2x^3+5x^2-12),$$

one has to consider the coefficients of

4 Functions

4.1 Definitions

Definition 4.1 (Functions)**.** *A functions is a mathematical relationship consisting of a rule linking elements from two sets such that each element from the first set (the domain) links to one and only one element from the second set (the image set or range).*

Remark 4.1. *The vertical line test allows to decide whether a graph represents a function: the graph represents a function if any line drawn parallel to the y-axis cuts the graph in only one point.*

Remark 4.2. *The range or image of a function is, by the definition above, the collection of all values $f(x)$ when x ranges through the domain of f. In a graph, where you draw $(x, y = f(x))$ in a xy-grid, the domain is the collection of all the x values and the range is the collection of all the y values.*

Examples of typical functions are
Functions can be classified by different means. We will use the following properties

Definition 4.2 (Properties of Functions)**.**
Let f be a real-valued function.

- *f is called strictly monotonically increasing if $f(x) > f(y)$ for all $x > y$ and strictly monotonically decreasing if $f(x) < f(y)$ for all $x > y$. Example: The function $f(x) = e^x$ is strictly increasing and $f(x) = e^{-x}$ is strictly decreasing.*
- *f is called even if $f(x) = f(-x)$ and* odd *if $f(-x) = -f(x)$. Example: The function $f(x) = \sin(x)$ is odd and $f(x) = \cos(x)$ is even.*
- *f is called continuous if you can draw it without taking the pen from the paper. In*

$f(x) =$	Domain/image	Special properties	HELM ref.
e^x	Domain: $\mathbb{R}$, image: $\mathbb{R}_{>0}$	Strictly increasing, convex	HELM 6
$\ln(x)$	Domain $\mathbb{R}_{>0}$, image: $\mathbb{R}$	Strictly increasing, concave	HELM 6
$\sum_{k=0}^{n} a_k x^k$	Domain: $\mathbb{R}$, image: $\mathbb{R}$ if n is odd.	Properties depend on the degree. Work out examples!	HELM 3
$\sin(x)$	Domain: $\mathbb{R}$, image: $[-1,1]$	Periodic with period 2π.	HELM 4
$\cos(x)$	Domain: $\mathbb{R}$, image: $[-1,1]$	Periodic with period 2π.	HELM 4
$\tan(x)$	Domain: $\bigcup_{k\in\mathbb{Z}}\left(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi\right)$, image: $[-1,1]$	Periodic with period π.	HELM 4

Table 2: Some typical functions.

particular, continuous functions can not *have jumps as in the function*

$$f(x) = \begin{cases} 1, x \leq 0 \\ 3, x > 0 \end{cases}$$

4.2 Operations on functions

We can add, subtract, multiply, and divide functions by point-wise definition, i.e.

$$\begin{aligned} f+g(x) &= f(x)+g(x), \\ fg(x) &= f(x)g(x), \quad \text{and} \\ \frac{f}{g}(x) &= \frac{f(x)}{g(x)}, \quad \text{where} \quad g(x) \neq 0. \end{aligned}$$

Another important operation is the composition: let $f : A \to B$ and $g : B \to C$ be functions. Then, $g \circ f(x) = g(f(x))$ is a function from A to C. Unless C is contained in A, the function $f \circ g$ is not defined.

Example 4.1. *Let $f(x) = e^x$ and $g(x) = x^2+1$. Both functions are defined on $\mathbb{R}$. We can consider*

$$\begin{aligned} f \circ g(x) = f(g(x)) = e^{x^2+1}, \\ g \circ f(x) = g(f(x)) = e^{2x}+1. \end{aligned}$$

That also shows that, in general, $f \circ g$ is not equal to $g \circ f$ should both be definable.

4.3 What is the inverse of a function?

Definition 4.3 (One-one function)**.** *A function f is called one-one if every element of the domain is linked to a unique element of the image, i.e. $f(x) = f(y)$ implies $x = y$.*

Remark 4.3. *The horizontal line test allows to decide whether a graph represents a one-one function. If a line drawn parallel to the x-axis cuts the graph only once, the represented function is one-one.*

Definition 4.4 (Inverse function). *Let f be a one-one function. Then, there exists a function g, called the inverse function to f, such that $g(f(x)) = f(g(x)) = x$. We denote the function g by f^{-1} (not to be confused with $1/f$).*

Function f	Inverse function f^{-1}	HELM ref.
$e^x : \mathbb{R} \to R_{>0}$	$\ln(x) : \mathbb{R}_{>0} \to \mathbb{R}$	HELM 3
$\sin(x) : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$	$\arcsin(x) : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	HELM 4
$\cos(x) : [0,\pi] \to [-1,1]$	$\arccos(x) : [-1,1] \to [0,\pi]$	HELM 4
$\tan(x) : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to (-\infty,+\infty)$	$\arctan(x) : (-\infty,+\infty) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	HELM 4

Table 3: Some typical functions with inverses.

4.4 How to compute the inverse?

Let us explain the computation of an inverse of a function by an example. Let f be given by

$$f(x) = 3x + 15.$$

First, one needs to make sure that the functions is one-one. The horizontal line test allows us to conclude, that f is one-one on its domain. We interchange x and y and solve the resulting equation for y:

$$y = f(x) = 3x + 15 \quad \Rightarrow \quad x = 3y + 15$$

which leads to

$$y = f^{-1}(x) = \frac{x}{3} - 15.$$

What if the horizontal line test fails, e.g. for $\sin(x)$? It may still be possible to invert the function on a subset of the domain. The strategy is to look for monotonicity intervals, i.e. parts of the domain in which the function is either strictly monotonically increasing or strictly monotonically decreasing.

5 Calculus for functions of one variable

The derivative af a sufficiently smooth (i.e. differentiable function) at a point x_0 of its domain is defined by

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

This is called the differential quotient. This is not to be confused with the difference quotient which is, for $x_0 < x_1$ in the domain fo f, usually defined as

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

where $y = f(x)$.

6 Matrices and linear systems of equations

A $m \times n$ matrix A is a rectangular array of numbers with m rows and n columns. We denote it by

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}.$$

The a_{ij} in the matrix are called **elements**. The index i in $a_i j$ says that a_{ij} sits in row i and the j says that it sits in column j. For **square matrices**, i.e. $m = n$, we denote the elements a_{ii} the diagonal elements and a matrix where only the a_{ii} are not all zero (or where $a_{ij} = 0$ whenever $i \neq j$) a diagonal matrix, e.g.

$$\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

A special diagonal matrix is the **identity matrix** $E_n = I_n = I$. This matrix has only zeros off the diagonal and 1 on the diagonal, i.e.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in the case $n = 3$. We also introduce the transpose A^T of a matrix A, which is the matrix where a_{ij} is replaced by a_{ji}, i.g.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix}.$$

Should we have that $A^T = A$, say that A is **symmetric**. Finally, if all elements bellow the diagonal are equal to 0, we call a matrix **upper triangular** and if all elements above the diagonal ore equal to 0, we call a matrix **lower triangular**.

Question 6.1. *Find examples for symmetric, upper triangular, and lower triangular matrices.*

Now, we can define arithmetic operations on matrices:

- Let $c \in \mathbb{R}$. Then, cA is the matrix where all a_{ij} are multiplied by c.
- Let A and B be $m \times n$ matrices. Then we define the sum of A and B by

$$A + B = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

  For addition, we have that $A + B = B + A$ (commutativity) as well as $(A + B) + C = A + (B + C)$ (associativity).

- Combining the two points above, $A - B$ is defined by $A + (-B)$.
- We can multiply matrices as well. The two matrices do not need to have the same dimensions. However, for $A \cdot B$ to make sense, the number of columns of A bust equal the number of rows of B. Thus, we can multiply a $m \times n$ matrix A by a $n \times k$ matrix B. For $B \cdot A$ to make sense, we additionally need $m = k$.

  To define it, we first define the following multiplication:

$$a \cdot b = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

  With that, we can define the product of two matrices by

$$\begin{bmatrix} - & a_1 & - \\ \vdots & \vdots & \vdots \\ - & a_m & - \end{bmatrix} \cdot \begin{bmatrix} | & \dots & | \\ b_1 & \dots & b_k \\ | & \dots & | \end{bmatrix} = \begin{bmatrix} a_1 \cdot b_1 & \dots & a_1 \cdot b_k \\ \vdots & \vdots & \vdots \\ a_m \cdot b_1 & \dots & a_m \cdot b_k \end{bmatrix},$$

  where the a_i in the first factor represent the ith row and the b_i in the second factor represent the ith column. Assuming that A, B, and C have the right proportions such that the expressions make sense, we have the following rules:

  - $A(B + C) = AB + AC$
  - $(B + C)A = BA + CA$
  - In general, we have $AB \neq BA$.
  - If $AB = 0$, we can not conclude that either A or B has to be zero, e.g.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

  .

Question 6.2. *Find examples of non-zero A and B such that $AB = 0$. Find examples of A and B such that AB and BA make sens but $AB \neq BA$.*

6.1 Determinants

Determinants are an important function which associate to every square matrix a real number. If the determinant of A, which we denote by $\det(A)$ or $|A|$, is not equal to 0, then the matrix A is invertible, in other words, there exists a matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = E_n.$$

We will be concerned with 2×2 and 3×3 matrices. In the lecture, we discussed how to get the following formulas:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{6.1}$$

and

$$
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \tag{6.2}
$$
$$
= aei + bfg + cdh - fha - bdi - ceg
$$

Question 6.3. *Write down all the other possibilities*[1] *to compute the determinant the above 3×3 matrix. Don't forget the chess-board rule. Just write down some 3×3 matrices and compute their determinant.*

Question 6.4. *As you can easily convince yourself, the determinant of a diagonal matrix is just the product of the diagonal elements. Convince yourselves by example, that the determinant of a triangular matrix is also given by the product of the diagonal elements.*

For 3×3 determinants, there is another way to compute them. This is the so-called Rule of Sarrus:

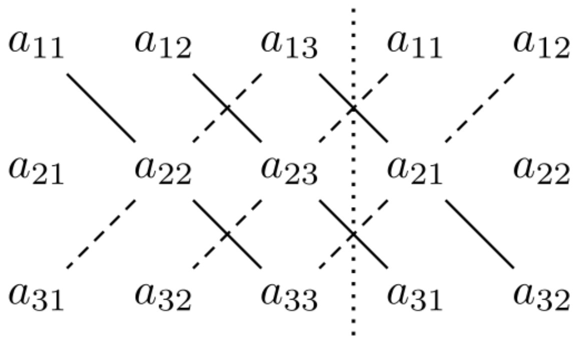

Figure 1: Sarrus Rule: The determinant of the three columns on the left is the sum of the products along the solid diagonals minus the sum of the products along the dashed diagonals.

Important facts to remember:

- If $\det(A) = 0$, the matrix a is not invertible. That means that the system $Ax = b$ may have no or infinitely many solutions.
- If $\det(A) \neq 0$, the matrix is invertible, i.e. there exists a matrix A^{-1} such that
$$AA^{-1} = A^{-1}A = E_n.$$
That also implies that the system $Ax = b$ has exactly one solution and it is given by $x = A^{-1}b$.
- The determinant can effectively be computed with the Gauss–algorithm: the following rules must be observed:
  - Switching two rows multiplies the determinant by -1.

[1] This is called Laplace Expansion.

– Multiplying any row by a non-zero real number multiplies the determinant by the same number.

– Adding a multiple of any row to any other row does not change the determinant.

Let us compute the determinant of a matrix using the Gauß-algorithm: consider

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 1 \\ 0 & -3 & 2 \end{bmatrix}.$$

First, we subtract two times the first line from the second. That does not change the determinant. We obtain

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ 0 & -3 & 2 \end{bmatrix}$$

Now, we add three times the second row to two times the third. Thus, the new matrix has determinant $2 \cdot \det(B)$. We obtain

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 13 \end{bmatrix}.$$

since this matrix is a triangular matrix, its determinant is given by the product of the diagonal elements $2 \cdot 13 = 26$. Thus, we have

$$2 \cdot \det(B) = 26 \quad \Leftrightarrow \quad \det(B) = 13.$$

6.1.1 Minors and cofactors

Given a matrix

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}.$$

Let M_{ij} be the determinant of the matrix that remains if one deletes the ith row and jth column from A. This is called the (i,j)-**minor**[2] of A. The number $A_{ij} = (-1)^{j+i} M_{ij}$ is called the **cofactor** of a_{ij} or (i,j)-cofactor.

Question 6.5. *Can you convince yourselves that the sum of the right hand side of* (6.2) *is a sum of minors/cofactors? Same question for* (6.1).

6.2 Cofactors and inverses

One can compute the inverse of a matrix A with the help of the cofactors[3]. This works as follows

[2] See also here. (Wikipedia)

[3] This is a quite ineffective method since it requires the computation of many determinants. If one uses the elementary method to compute determinants, the workload is enormous (around $n!$ operations) and too much even for the fastest computers to handle.

1. Compute the cofactor matrix[4] $\text{cof}(A)$ as

$$\text{cof}(A) = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mn} \end{bmatrix},$$

   where the A_{ij} are the cofactors of a_{ij}.

2. Write down $\text{cof}(A)^T$ and compute the determinant $\det(A)$.

3. Write down the inverse of A as

$$A^{-1} = \frac{1}{\det(A)} \text{cof}(A)^T.$$

Remark 6.1. *You also may have a look at the leaflet of the mathcentre.*

Let us do an example. We compute the inverse of

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 1 \\ 0 & -3 & 2 \end{bmatrix}.$$

We get the cofactors

$$B_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 1 \\ -3 & 2 \end{vmatrix}, \quad B_{21} = (-1)^{2+1} \begin{vmatrix} 0 & -1 \\ -3 & 2 \end{vmatrix}$$
$$B_{31} = (-1)^{3+1} \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix}, \quad B_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}$$
$$B_{22} = (-1)^{2+2} \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}, \quad B_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$
$$B_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 2 \\ 0 & -3 \end{vmatrix}, \quad B_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix}$$
$$B_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}.$$

As you will convince yourselves easily, we obtain

$$\text{cof}(A) = \begin{bmatrix} 7 & -4 & -6 \\ 3 & 2 & 3 \\ 2 & -3 & 2 \end{bmatrix}.$$

With $\det(B) = 13$, we obtain

$$B^{-1} = \frac{1}{13} \begin{bmatrix} 7 & 3 & 2 \\ -4 & 2 & -3 \\ -6 & 3 & 2 \end{bmatrix}.$$

[4] In HELM 7.4 you can find this algorithm too, However, there they compute the adjoint which is the transpose of the cofactor matrix. The name adjoint is somewhat misleading as that is usually A^T. The matrix computed there should be called classical adjoint or adjunct.

7 The Gauss–algorithm and inverses

Given systems of linear equations, the Gauss–algorithm[5] is a way to get to a solution by reducing the system to triangular form which then allows to solve the system recursively. The operations one can use to reduce the matrix to triangular form are:

1. Swap the positions of two rows.
2. Multiply a row by a non-zero scalar.
3. Add to one row a scalar multiple of another.

Let us perform a simple example. Consider

$$\begin{array}{rrcrcr} & x_1 & + & 2x_2 & = & 3 \\ 3 & x_1 & + & 2x_2 & = & 5 \end{array}.$$

To solve the system, we work with the augmented coefficient matrix to reduce the necessary writing: the augmented coefficient matrix is

$$\left.\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 2 & 5 \end{array}\right..$$

To solve the system, we subtract three times the first row from the second and obtain

$$\left.\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -4 & -4 \end{array}\right..$$

The last equation $-4x_2 = -4$ gives $x_2 = 1$. Substituting this in the first equation, we get

$$x_1 + 2 \cdot 1 = 3 \quad \Leftrightarrow \quad x_1 = 1.$$

Let us now discuss possible cases that may occur solving linear systems. Consider

$$\begin{array}{rrcrcr} & x_1 & + & 2x_2 & = & 3 \\ 3 & x_1 & + & ax_2 & = & 5 \end{array}.$$

Again, we subtract three times the first row from the second and obtain

$$\left.\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & a-6 & -4 \end{array}\right..$$

Now, the last equation reads as

$$(a-6)x_2 = -4.$$

Dividing by $a-6$, we obtain

$$x_2 = -\frac{4}{a-6}$$

and, plugging that into the first equation, we obtain

$$x_1 + 2 \cdot \left(-\frac{4}{a-6}\right) = 3 \quad \Leftrightarrow \quad x_1 = 3 + \frac{8}{a-6} = \frac{3a-10}{a-6}.$$

[5]See also here. (Wikipedia)

When are we allowed to do all that? Since we where dividing by $a-6$, we need that this number is not equal to zero. Thus, we have a unique solution of $a-6\neq 0$, i.e. $a\neq 6$. If $a=6$, we get

$$0\cdot x_2 = -4$$

which has no solutions.
Let us now modify the question a little bit further. Consider

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 3 \\ 3\ x_1 & + & ax_2 & = & b \end{array}.$$

Again subtracting three times the first row from the second, we obtain

$$\left.\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & a-6 & b-9 \end{array}\right. .$$

Now, we have three cases to consider since the last line reads as

$$(a-6)x_2 = b-9.$$

If $a-6\neq 0$ (i.e. $a\neq 6$), we can divide by $a-6$ and obtain

$$x_2 = \frac{b-9}{a-6}, \quad \to \quad x_1 = 3 - \frac{2(b-9)}{a-6}.$$

Thus, in this case we have exactly one solution. If $a=6$, we have the equation

$$0\cdot x_2 = b-9.$$

It has infinitely many solutions if $b=9$ and no solution if $b\neq 9$. Putting everything together, we obtain

- Infinitely many solutions: $a=6$, $b=9$.
- No solutions: $a=6$, $b\neq 9$.
- Exactly one solution: $a\neq 6$.

More examples for systems with three unknowns can be found in Section 7.2.

7.1 Computing inverses with the Gauss–algorithm

To compute an inverse, the algorithm[6][7] works as follows

1. Write down $A|E$.
2. Use the Gauss–algorithm[8] to produce the identity matrix on the left hand side. The allowed operations in the Gauss–algorithm[9] are:

[6] See also here. (Wikipedia)
[7] You can also find that in HELM 7.3.3.
[8] See also HELM 8.3.
[9] To be more precise, this version is usually called Gauss–Jordan algorithm.

- Swap the positions of two rows.
- Multiply a row by a non-zero scalar.
- Add to one row a scalar multiple of another.

Remark 7.1. *A fact to remember is that a system can never have a unique solution if the coefficient matrix has two identical rows or columns. You should check by example, that the determinant of the coefficient matrix is equal to zero in that case. Also, if the Gauss–algorithm arrives at a zero line in the augmented system, one variable can be chosen freely, say* $s \in \mathbb{R}$.

Remark 7.2. *There are several explanations and examples concerning the Gaußalgorithm on* ***youtube*** *that might be helpful for you: patrickJMT 1, patrickJMT 2, MIT OpenCourseWare. Here, two showing the computation of an inverse with the Gauß–algorithm: patrickJMT 1 3, MIT OpenCourseWare.*

As an example, let us compute the inverse of

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 1 \\ 0 & -3 & 2 \end{bmatrix}.$$

First, let us compute the determinant to see that B has an inverse:

$$\begin{vmatrix} 1 & 0 & -1 \\ 2 & 2 & 1 \\ 0 & -3 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 2 & 2 \\ 0 & -3 \end{vmatrix} = 13,$$

where we chose the first row to develop the determinant. Now we use the Gauss-algorithm to compute the inverse of the matrix B[10]:

$$\begin{array}{ccc|ccc||l}
1 & 0 & -1 & 1 & 0 & 0 & \\
2 & 2 & 1 & 0 & 1 & 0 & \\
0 & -3 & 2 & 0 & 0 & 1 & \\
\hline
1 & 0 & -1 & 1 & 0 & 0 & \\
0 & 2 & 3 & -2 & 1 & 0 & = R_2 - 2R_1 \\
0 & -3 & 2 & 0 & 0 & 1 & \\
\hline
1 & 0 & -1 & 1 & 0 & 0 & \\
0 & 2 & 3 & -2 & 1 & 0 & \\
0 & 0 & 13 & -6 & 3 & 2 & = 2R_3 + 3R_2 \\
\hline
1 & 0 & 0 & \frac{7}{13} & \frac{3}{13} & \frac{2}{13} & = R_1 + \frac{1}{13}R_3 \\
0 & 2 & 0 & -\frac{8}{13} & \frac{4}{13} & -\frac{6}{13} & \\
0 & 0 & 13 & -6 & 3 & 2 & = 2R_3 + 3R_2
\end{array}.$$

From that, we obtain

$$B^{-1} = \frac{1}{13} \begin{bmatrix} 7 & 3 & 2 \\ -4 & 2 & -3 \\ -6 & 3 & 2 \end{bmatrix}.$$

[10]The algorithm that we introduced is solving three systems simultaneously: $Bx = e_1$, $Bx = e_2$, and $Bx = e_3$. The inverse is the matrix with the three resulting vectors as columns.

7.2 Systems with parameters

This section is a continuation of the considerations at the very beginning of this section. Here, we consider systems with parameters and three variables. To start, you may set the parameters to some numbers, do the calculations and only then read further in the solution presented here. Also note that there is usually more than one way to arrive at the correct conclusion. You can only become good in solving systems of this type if you practice enough.

7.2.1 First example

Consider the system $Ax = b$ given by

$$\begin{array}{ccccccl} & & 3x_2 & + & 6x_3 & = 9 \\ x_1 & + & 4x_2 & + & 7x_3 & = 8 \\ 2x_1 & + & 5x_2 & + & \alpha x_3 & = \beta \end{array} .$$

Let us sole the system and find conditions on α and β such that the system has (i) no solutions, (ii) exactly one solution, (iii) infinitely many solutions. First, the Augmented coefficient matrix is

$$\left.\begin{array}{ccc|c} 0 & 3 & 6 & 9 \\ 1 & 4 & 7 & 8 \\ 2 & 5 & \alpha & \beta \end{array}\right. .$$

Using the Gauss–algorithm, we obtain

$$\begin{array}{cccc|c} 0 & 3 & 6 & 9 \\ \boxed{1} & 4 & 7 & 8 \\ 2 & 5 & \alpha & \beta \\ \hline 0 & 3 & 6 & 9 \\ 1 & 4 & 7 & 8 \\ 0 & \boxed{\text{-}3} & \alpha - 14 & \beta - 16 \\ \hline 0 & 0 & \alpha - 8 & \beta - 7 \\ 1 & 4 & 7 & 8 \\ 0 & -3 & \alpha - 14 & \beta - 16 \end{array}$$

From that we get the following cases:

- infinitely many solutions for $\alpha = 8$, $\beta = 7$,
- exactly one solution for $\alpha \neq 8$, and
- no solution for $\alpha = 8$, $\beta \neq 7$.

7.2.2 Second example

Now, consider the system $Ax = b$ given by

$$\begin{array}{ccccccl} x_1 & - & 2x_2 & + & \alpha x_3 & = 2 \\ x_1 & + & x_2 & + & x_3 & = 2 \\ -2x_1 & - & 3x_2 & - & x_3 & = \beta \end{array} .$$

Using the Gauss–algorithm, we compute

$$
\begin{array}{ccc|c}
1 & -2 & \alpha & 2 \\
\boxed{1} & 1 & 1 & 2 \\
-2 & -3 & -1 & \beta \\
\hline
0 & -3 & \alpha-1 & 0 \\
1 & 1 & 1 & 2 \\
0 & \boxed{-1} & 1 & \beta+4 \\
\hline
0 & 0 & \alpha-4 & -3(\beta+4) \\
1 & 1 & 1 & 2 \\
0 & -1 & 1 & \beta+4
\end{array}
$$

From that we get the following cases:

- infinitely many solutions for $\alpha = 4$, $\beta = -4$,
- exactly one solution for $\alpha \neq 4$, and
- no solution for $\alpha = 4$, $\beta \neq -4$.

7.2.3 Third example

Now, consider the system $Ax = b$ given by

$$
\begin{array}{rcrcrl}
3x_1 & + & 2x_2 & - & 4x_3 & = -1 \\
-2x_1 & - & x_2 & + & 3x_3 & = 1 \\
2x_1 & & & + & \alpha x_3 & = \beta
\end{array} .
$$

Using the Gauss–algorithm, we compute

$$
\begin{array}{ccc|c}
3 & 2 & -4 & -1 \\
-2 & \boxed{-1} & 3 & 1 \\
2 & 0 & \alpha & \beta \\
\hline
\boxed{-1} & 0 & 2 & 1 \\
-2 & -1 & 3 & 1 \\
2 & 0 & \alpha & \beta \\
\hline
-1 & 0 & 2 & 1 \\
-2 & -1 & 3 & 1 \\
0 & 0 & \alpha+4 & \beta+2
\end{array}
$$

From that we get the following cases:

- infinitely many solutions for $\alpha = -4$, $\beta = -2$,
- exactly one solution for $\alpha \neq -4$, and
- no solution for $\alpha = -4$, $\beta \neq -2$.